Classification of multidimensional inflationary models

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Abstract

We define under which circumstances two multi-warped product spacetimes can be considered equivalent and then we classify the spaces of constant curvature in the Euclidean and Lorentzian signature. For dimension $D = 2$, we get essentially twelve representations, for $D = 3$ exactly eighteen. More general, for every even D, $5D + 2$ cases exist, whereas for every odd D, $5D + 3$ cases exist. For every D, exactly one half of them has the Euclidean signature. Our definition is well suited for the discussion of multidimensional cosmological models, and our results give a simple algorithm to decide whether a given metric represents the inflationary de Sitter spacetime (in unusual coordinates) or not.

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I. Introduction.

Recently, a lot of papers dealt with multi-dimensional cosmological models, cf. e.g. [1,2,3] and references cited there. In these works, the ansatz for the metric is a generalized warped product of several spaces, with the warping functions depending on the time t only, i.e.

$$ds^2 = dt^2 - \sum_{k=1}^{n} a_k^2(t)\ d\sigma_k^2 \tag{1.1}$$

where each $d\sigma_k^2$ represents a Riemannian space of dimension $d_k \geq 1$. In the usual interpretation, one requires $d_1 = 3$ and $d\sigma_1^2$ to be the physical 3-space, whereas all other spaces $d\sigma_k^2$, $k \geq 2$ are internal spaces. However, our approach is not restricted to this interpretation.

One of the often discussed questions in this context is the appearence of an inflationary phase of cosmic evolution. Its geometry is usually represented by a spacetime of constant negative curvature, called de Sitter spacetime. It is well-known how to represent the de Sitter spacetime of arbitrary dimension in the form (1.1), but up to now there does not exist a classification of the metrics of the general form (1.1) to decide which of them correspond to spaces of constant curvature.

To elucidate what this means, let us consider the metrics (1.2) and (1.3).

$$ds^2 = dt^2 - \sinh^2 t\ dx^2 - \cosh^2 t\ (dy^2 + \sin^2 y\ dz^2) \tag{1.2}$$

(see e.g. [4, eq.(3.18)]). The metric (1.2) is of the type (1.1) and is obviously a cosmological model of Kantowski-Sachs type. The metric (1.2) is known to represent a vacuum solution of Einstein's equations with a positive cosmological term. In agreement with the cosmological no-hair theorem it is asymptotically de Sitter. In the coordinates used, the metric (1.2) seems to define an anisotropic model, however (cf. [4]) it is nothing but a piece of the isotropic de Sitter spacetime in unusual coordinates, and moreover, the coordinate transformation necessary to bring it to the classical form is quite involved.

A related metric is (cf. [5, eq. (4.6)]),

$$ds^2 = dt^2 + \sin^2 t\ dx^2 + \cos^2 t\ dy^2 \tag{1.3}$$

which is the metric of the standard 3-sphere in unusual coordinates.

It is the purpose of the present paper to give a complete classification of all the metrics of type (1.1) which represent spaces of constant curvature.

The paper is organized as follows: sect. II introduces the generalized warped products and defines the normal form of the metric (1.1). Sect. III gives the results for the dimension $D \le 3$ and sect. IV covers the remaining cases $D \ge 4$. In sect. V, we consider a slightly different form of the metric, where the a_k can additionally depend on one of the spatial coordinates, but the $d\sigma_k^2$ have dimension 1. Sect. VI discusses the results.

II. Generalized warped products.

Usually, a warped product is the conformally transformed cartesian product between two Riemannian spaces, where the conformal factor, called warping function, depends on the coordinates of one of the factors only. If there are more than two factor spaces, then there exist several possible generalizations of that notion. We will restrict ourselves to the following types of generalized warped products: all $a_k > 0$ and

$$ds^2 = dt^2 \pm \sum_{k=1}^{n} a_k^2(t)\ d\sigma_k^2 \tag{2.1}$$

and each $d\sigma_k^2$ is a Riemannian space of dimension $d_k \ge 1$. Thus we consider the Euclidean (upper sign) and the Lorentzian (lower sign) signatures of the metric. In this way we are able to cover both eqs. (1.2) and (1.3). Obviously, the metric (2.1) represents a usual warped product for $n = 1$. The dimension of ds^2 is $D = 1 + \sum d_k$. We want to find out under which circumstances the metric (2.1) represents a space of constant curvature locally (so we do not discuss whether zeroes of the functions $a_k(t)$ give rise to coordinate singularities and whether our metrics are geodesically complete).

To get a uniquely defined classification, we start by explaining under which circumstances two metrics of type (2.1) will be considered to be the same:

A) Any constant $c_k > 0$ can be shifted between $a_k^2(t)$ and $d\sigma_k^2$;

B) ds^2 can be multiplied by a positive constant, or in other words: t and every $a_k(t)$ can be multiplied by the same positive factor;

C) Linear transformations of t will be allowed;

D) Permutations of the indices k are allowed;

E) Coordinate transformations involving only the coordinates of one of the spaces $d\sigma_k^2$ are allowed.

F) If two warping functions coincide, $a_j(t) = a_k(t)$, $(j \neq k)$, then $a_j^2(t)d\sigma_j^2 + a_k^2(t)d\sigma_k^2$ can be replaced by $a_j^2(t)(d\sigma_j^2 + d\sigma_k^2)$.

Comment: B) means that we do not distinguish between homothetically equivalent spaces, or, in other words, a change of the length unit will not be considered essential. F) means that the number n of terms in the sum of eq. (2.1) is not fixed. However, we can define the normal form of the representation (2.1) as follows: whenever possible (if necessary by the help of introducing suitable constants c_k according to A), the replacement F) shall be done (of course, this procedure terminates uniquely). Equivalently we may say: the normal form of the representation is that one where the number n is minimal. Furthermore it holds: the metric (2.1) is in normal form if the quotients $a_j(t)/a_k(t)$ are constant for $k = j$ only.

For technical reasons, we will sometimes discuss spaces in non-normal form. However, if the number n of factors is explicitly mentioned, we always refer this to the normal form and, without loss of generality, we may restrict to metrics (2.1) in normal form.

III. Classification for dimension $D \leq 3$.

$D = 1$ implies $n = 0$ and $ds^2 = dt^2$ represents flat space. So we have just one representation for $D = 1$.

III.1. Dimension $D = 2$.

Let $D = 2$, then $d\sigma_1^2 = dx^2$ is flat[1].

We write $a(t)$ instead of $a_1(t)$ and get

$$ds^2 = dt^2 \pm a^2(t)\, dx^2 \tag{3.1}$$

which is already in normal form. This space is of constant curvature iff the curvature scalar $R = -2\ddot{a}/a$ is constant, where a dot denotes d/dt. Both

[1] Spaces of constant curvature in $D = 2$ have been discussed, for example, in [6].

signatures can be considered simultaneously, because they are related by the imaginary coordinate transformation $x \to ix$. (Here we give the derivation for the known case $D = 2$ in detail in order to show how our formulation works, we do not repeat the analogous arguments in the other cases). According to B), we may restrict to the three cases $R = 0$, $R = 2$ and $R = -2$. For $R = 0$, $a(t)$ must be linear in t, and according to A) and C), the following representations are singled out:

$$ds^2 = dt^2 \pm dx^2 \tag{3.2}$$

and

$$ds^2 = dt^2 \pm t^2 dx^2 \tag{3.3}$$

Clearly, these are the cartesian and polar coordinates respectively for the flat space. For $R = 2$, $\ddot{a} + a = 0$ has to be solved. According to A) and C), only one representation, the standard metric of the 2-sphere, appears:

$$ds^2 = dt^2 \pm \sin^2 t \; dx^2 \tag{3.4}$$

For $R = -2$, however, three different representations appear for each of the two signatures:[2]

$$ds^2 = dt^2 \pm e^{2t} \; dx^2 \tag{3.5}$$

further

$$ds^2 = dt^2 \pm \sinh^2 t \; dx^2 \tag{3.6}$$

and

$$ds^2 = dt^2 \pm \cosh^2 t \; dx^2 \tag{3.7}$$

An explicit coordinate transformation between the metrics (3.5) and (3.6) is deduced in the appendix of ref. [4]. Metric (3.4) and (3.6) are related by multiplying s, t and x by i.

Thus we have exactly twelve different representations of a space of constant curvature for $D = 2$. Six of them have $R < 0$ and only two have $R > 0$.

III.2. Dimension $D = 3$. First part.

[2]By condition C) the solution with e^{-2t} instead of e^{2t} can be excluded.

Let $D = 3$. The space is of constant curvature iff the three eigenvalues of the Ricci tensor coincide. Let us start recalling the known cases. The cartesian product of flat spaces is flat. So we can get from eq. (3.2), (3.3) for the $R = 0$ cases:

$$n = 1, \qquad ds^2 = dt^2 \pm (dx^2 + dy^2) \tag{3.8}$$

and

$$n = 2, \qquad ds^2 = dt^2 \pm (t^2 \ dx^2 + dy^2) \tag{3.9}$$

For $R > 0$ we know the examples

$$n = 1, \qquad ds^2 = dt^2 + \sin^2 t \ (dx^2 + \sin^2 x \ dy^2) \tag{3.10}$$

and

$$n = 2, \qquad ds^2 = dt^2 \pm (\sin^2 t \ dx^2 + \cos^2 t \ dy^2) \tag{3.11}$$

Here, (3.10) is the standard three-sphere and (3.11) is the metric (1.3) [5]. Replacing x by ix, we get from the metric (3.10) the Lorentzian signature with $R > 0$:

$$n = 1, \qquad ds^2 = dt^2 - \sin^2 t \ (dx^2 + \sinh^2 x \ dy^2) \tag{3.12}$$

For $R < 0$ we get analogously, as in sect. III.1:

$$n = 1, \qquad ds^2 = dt^2 \pm e^{2t} \ (dx^2 + dy^2) \tag{3.13}$$

and

$$n = 2, \qquad ds^2 = dt^2 \pm (\sinh^2 t \ dx^2 + \cosh^2 t \ dy^2) \tag{3.14}$$

(cf. [5, eq.(3.9)]). So, we have again twelve different cases. However, before checking their completeness in section III.4, we have to discuss the case $n = 1$ in the next section.

III.3. The case $n = 1$ in arbitrary dimension.

Let $D \geq 3$ be arbitrary and be $n = 1$ in eq.(2.1). So the index 1 may be omitted and $d = D - 1$. We write $d\sigma^2 = h_{\alpha\beta} dx^\alpha dx^\beta$, where $h_{\alpha\beta}$ depends on the spatial coordinates only, $\alpha, \beta = 1, ..., d$. For $ds^2 = dt^2 \pm a^2(t) d\sigma^2$ we get

$R_{00} = -d\ddot{a}/a$. Indices $j, k, ...$ take the D values $0, 1, ..., d$. If ds^2 is a space of constant curvature, then it must be an Einstein space, i.e. Ricci tensor and metric tensor differ by a constant factor only. So we may assume, without loss of generality due to condition B), that

$$R_{jk} = \lambda d g_{jk}, \qquad \text{with } \lambda \in \{-1, 1, 0\} \tag{3.15}$$

Now, $g_{00} = 1$ and we set $\lambda = -\ddot{a}/a$. So we are in a case similar to that of sect. III.1. Consequently, only the same six functions $a(t)$ may appear: $a = 1$ and $a = t$ for $R = 0$, $a = \sin t$ for $R > 0$, and $a = e^t$, $a = \sinh t$, $a = \cosh t$ for $R < 0$.

Let us now apply the $\alpha\beta$-component of eq. (3.15), and let $P_{\alpha\beta}$ denote the Ricci tensor of $d\sigma^2$. We get:

$$P_{\alpha\beta} = R_{\alpha\beta} \pm h_{\alpha\beta}[a\ddot{a} + (d-1)\dot{a}^2] \tag{3.16}$$

From (3.15) and (3.16) we get

$$P_{\alpha\beta} = \Lambda h_{\alpha\beta}, \qquad \Lambda = \pm(d-1)(\dot{a}^2 + \lambda a^2) \tag{3.17}$$

It is easy to see that Λ is a constant. Concluding, for $n = 1$, $d\sigma^2$ must be an Einstein space of constant curvature scalar.

III.4. Dimension $D = 3$. Second part.

Now we are ready to complete the discussion for $D = 3$. For $n = 1$, we have $d = 2$, and by sect. III.3, $d\sigma^2$ has constant curvature scalar. This means that $d\sigma^2$ is of constant curvature here, and (3.17) reads $\dot{a}^2 + \lambda a^2 = \pm\Lambda$. $\lambda = 0$, $a = 1$ gives the metric (3.8). $\lambda = 0$, $a = t$, gives $\Lambda = \pm 1$, namely

$$n = 1, \qquad ds^2 = dt^2 + t^2(dx^2 + \sin^2 x \; dy^2) \tag{3.18}$$

which is flat space in spherical coordinates , and

$$n = 1, \qquad ds^2 = dt^2 - t^2(dx^2 + \sinh^2 x \; dy^2) \tag{3.19}$$

$\lambda = 1$, $a = \sin t$ gives $\Lambda = \pm 1$, i.e. the metric (3.10) and (3.12). $\lambda = -1$, $a = e^t$ yields $\Lambda = 0$, i.e. the metric (3.13). $\lambda = -1$, $a = \sinh t$ yields $\Lambda = \pm 1$, i.e.,

$$n = 1, \qquad ds^2 = dt^2 + \sinh^2 t \; (dx^2 + \sin^2 x \; dy^2) \tag{3.20}$$

and

$$n = 1, \qquad ds^2 = dt^2 - \sinh^2 t \; (dx^2 + \sinh^2 x \; dy^2) \tag{3.21}$$

Finally, $\lambda = -1$, $a = \cosh t$ yields $\Lambda = \mp 1$, i.e.

$$n = 1, \qquad ds^2 = dt^2 + \cosh^2 t \; (dx^2 + \sinh^2 x \; dy^2) \tag{3.22}$$

and

$$n = 1, \qquad ds^2 = dt^2 - \cosh^2 t \; (dx^2 + \sin^2 x \; dy^2) \tag{3.23}$$

This completes the discussion for $n = 1$, and the six metrics (3.18)-(3.23) together with the twelve metrics of section III.2 form already eighteen different representations of a space of constant curvature in $D = 3$. It remains to check completeness for the case $n = 2$. So we have to use the ansatz

$$ds^2 = dt^2 \pm (a^2(t) \; dx^2 + b^2(t) \; dy^2) \tag{3.24}$$

and have to ensure that the three eigenvalues of the Ricci tensor of (3.24) coincide. This is equivalent to

$$\frac{\ddot{a}}{a} = \frac{\ddot{b}}{b} = \frac{\dot{a}}{a}\frac{\dot{b}}{b} \tag{3.25}$$

A detailed investigation of eq. (3.25) shows that all solutions are already covered by the eighteen cases mentioned before.

Let us summarize the results for $D = 3$: six representations with $n = 2$ and twelve representations with $n = 1$ form a complete classification. Six of these are flat, four have positive curvature and the remaining eight have $R < 0$.

IV. Classification for dimension $D \geq 4$.

From now on, we put $D \geq 4$. It holds: ds^2 is of constant curvature iff it is an Einstein space with vanishing Weyl tensor. In sect. IV.1 we review the known cases, and in sect. IV.2 we give a general account of conformal flatness.

IV.1. Dimension $D \geq 4$. First part.

Let $\delta_{\alpha\beta}$ be the Kronecker tensor. The generalization of eqs. (3.8), (3.9) with $R = 0$ reads

$$n = 1, \qquad ds^2 = dt^2 \pm \delta_{\alpha\beta} dx^\alpha dx^\beta \tag{4.1}$$

and

$$n = 2, \qquad ds^2 = dt^2 \pm (t^2 \; dx^2 + \delta_{\alpha\beta} dx^\alpha dx^\beta) \tag{4.2}$$

Generalizing (3.18), (3.19) with $R = 0$ yields

$$n = 1, \qquad ds^2 = dt^2 + t^2 d\Omega_k^2 \tag{4.3}$$

and

$$n = 1, \qquad ds^2 = dt^2 - t^2 d\bar{\Omega}_k^2 \tag{4.4}$$

where $k = D - 1$ and $d\Omega_k^2$ is the metric of the standard sphere of dimension k, while $d\bar{\Omega}_k^2$ denotes the corresponding space of constant negative curvature.

For $R > 0$ we get from (3.10) and (3.12)

$$n = 1, \qquad ds^2 = dt^2 + \sin^2 t \; d\Omega_k^2 \tag{4.5}$$

and

$$n = 1, \qquad ds^2 = dt^2 - \sin^2 t \; d\bar{\Omega}_k^2 \tag{4.6}$$

Eq.(4.6) is the anti-de Sitter spacetime written as an open Friedman model for $D = 4$.

For $R < 0$, we get from (3.13)

$$n = 1, \qquad ds^2 = dt^2 \pm e^{2t} \delta_{\alpha\beta} dx^\alpha dx^\beta \tag{4.7}$$

and from (3.20)-(3.23)

$$n = 1, \qquad ds^2 = dt^2 + \sinh^2 t \; d\Omega_k^2 \tag{4.8}$$

and

$$n = 1, \qquad ds^2 = dt^2 - \sinh^2 t \; d\bar{\Omega}_k^2 \tag{4.9}$$

further

$$n = 1, \qquad ds^2 = dt^2 + \cosh^2 t \; d\bar{\Omega}_k^2 \tag{4.10}$$

and

$$n = 1, \qquad ds^2 = dt^2 - \cosh^2 t \; d\Omega_k^2 \tag{4.11}$$

The metric (4.11) represents the de Sitter spacetime as a closed Friedman model for $D = 4$. So, for every $D \geq 4$, we have found twelve representations with $n = 1$, namely eqs. (4.1) and (4.3)-(4.11).

A further generalization of (4.3)-(4.4) is possible with $R = 0$ and $2 \leq k \leq D - 2$

$$n = 2, \qquad ds^2 = dt^2 + t^2 \; d\Omega_k^2 + \delta_{\alpha\beta} dx^\alpha dx^\beta \tag{4.12}$$

and

$$n = 2, \qquad ds^2 = dt^2 - t^2 \; d\bar{\Omega}_k^2 - \delta_{\alpha\beta} dx^\alpha dx^\beta \tag{4.13}$$

Together with (4.2) this yields $2(D-2)$ representationwith $R = 0$, $n = 2$.

For $D = 4$, the following representations are known, cf. eq. (1.2) above and ref. [4] for details.

$R > 0$

$$n = 2, \qquad ds^2 = dt^2 + \sin^2 t \; dx^2 + \cos^2 t \; d\Omega_2^2 \tag{4.14}$$

and

$$n = 2, \qquad ds^2 = dt^2 - \sin^2 t \; dx^2 - \cos^2 t \; d\bar{\Omega}_2^2 \tag{4.15}$$

$R < 0$

$$n = 2, \qquad ds^2 = dt^2 + \sinh^2 t \; dx^2 + \cosh^2 t \; d\bar{\Omega}_2^2 \tag{4.16}$$

and

$$n = 2, \qquad ds^2 = dt^2 - \sinh^2 t \; dx^2 - \cosh^2 t \; d\Omega_2^2 \tag{4.17}$$

further

$$n = 2, \qquad ds^2 = dt^2 + \cosh^2 t \; dx^2 + \sinh^2 t \; d\Omega_2^2 \tag{4.18}$$

and

$$n = 2, \qquad ds^2 = dt^2 - \cosh^2 t \; dx^2 - \sinh^2 t \; d\bar{\Omega}_2^2 \tag{4.19}$$

So, for $D = 4$, we have already 10 representations with $n = 2$. By the way, the metric (4.17) is the same as (1.2).

IV.2. Conformal flatness.

Spaces of constant curvature are conformally flat, and for $D \geq 4$, conformal flatness is equivalent to the vanishing of the conformally invariant Weyl tensor $C^i{}_{jkl}$, defined as

$$\begin{aligned} C_{ijkl} &= R_{ijkl} - \frac{1}{D-2}(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{jk}g_{il} - R_{il}g_{jk}) \\ &+ \frac{R}{(D-1)(D-2)}(g_{ik}g_{jl} - g_{jk}g_{il}) \end{aligned} \tag{4.20}$$

Now, let $n = 1$, i.e. $ds^2 = dt^2 \pm a^2(t)d\sigma^2$. According to sect. III.3, $d\sigma^2$ must be an Einstein space. But ds^2 is conformally flat, hence $d\hat{s}^2 = ds^2/a^2(t)$ is conformally flat too. After a time-rescaling, we set

$$d\hat{s}^2 = d\hat{t}^2 \pm d\sigma^2$$

Inserting this metric into eq. (4.20), applying $\hat{C}_{ijkl} = 0$ and the fact that $d\sigma^2$ is an Einstein space, gives as a result that $d\sigma^2$ must be a space of constant curvature . This proves that our classification for the case $n = 1$ given in sect. IV.1 is already complete.

Let $n \geq 2$ in the following. We want to show that every $d\sigma_k^2$ is a space of constant curvature . For $d_k = 1$, this is trivial. For any $d_k \geq 2$ we denote d_k by d and $D - d$ by m. Because of $n \geq 2$ we have $m \geq 2$. Let $d\sigma_k^2 = h_{\alpha\beta}(x^\alpha)dx^\alpha dx^\beta$ with $\alpha, \beta = 1, ..., d$, and we write

$$ds^2 = \pm a_k^2(t)d\sigma_k^2 + d\Omega^2$$

where $d\Omega^2$ is simply a description of the rest. Let $d\hat{s}^2 = \pm ds^2/a_k^2(t)$, then we may write

$$d\hat{s}^2 = h_{\alpha\beta}(x^\alpha)dx^\alpha dx^\beta + h_{AB}(x^A)dx^A dx^B$$

where $A, B = 0, d+1, ..., D-1$ (h_{AB} has unspecified signature). $d\sigma_k^2$ has curvature scalar $R_{(1)}$ and $d\hat{\Omega}^2 = h_{AB}dx^A dx^B$ curvature scalar $R_{(2)}$. Therefore, the metric $d\hat{s}^2$ has curvature scalar $\hat{R} = R_{(1)} + R_{(2)}$. Then we apply (4.20) and the fact that $C_{\alpha A\beta B}$ vanishes. This gives

$$R_{\alpha\beta}h_{AB} + R_{AB}h_{\alpha\beta} = \frac{\hat{R}}{D-1}h_{\alpha\beta}h_{AB} \tag{4.21}$$

We calculate the trace of this equation by multiplying with $h^{\alpha\beta}h^{AB}$. Observing that $d \geq 2$ and $m \geq 2$ this leads to

$$\frac{R_{(1)}}{d(d-1)} + \frac{R_{(2)}}{m(m-1)} = 0 \tag{4.22}$$

$R_{(1)}$ depends on x^α only, $R_{(2)}$ on x^A only, consequently both of them are constant. By the way, for $m = d = D/2$ we get additionally $\hat{R} = 0$. Knowing the constancy of $R_{(1)}$, we multiply eq. (4.21) by h^{AB} and get the result that $d\sigma_k^2$ is an Einstein space with constant curvature scalar. In the last step we apply $C_{\alpha\beta\gamma\delta} = 0$ with eq. (4.20) to show that $d\sigma_k^2$ is a space of constant curvature.

IV.3. Dimension $D \geq 4$. Second part.

According to the previous results, it remains to discuss the cases with

$$n \geq 2, \qquad ds^2 = dt^2 \pm \sum_{k=1}^{n} a_k^2(t) d\sigma_k^2 \tag{4.23}$$

where each of the spaces $d\sigma_k^2$ represents a space of constant curvature, and the quotients $a_k(t)/a_j(t)$ are constant for $k = j$ only. Let us start with the simplest case: every $d_k = 1$, i.e., $n = D - 1$, and we have to check the positive signature case only

$$n \geq 3, \qquad ds^2 = dt^2 + \sum_{k=1}^{n} a_k^2(t)(dx^k)^2 \tag{4.24}$$

We define $\alpha_k = \dot{a}_k/a_k$, $H = \sum \alpha_k$, $H_2 = \sum \alpha_k^2$, and $\beta_k = \alpha_k - H/n$. Consequently, $\sum \beta_k = 0$. The equation $R_{AB} = \lambda g_{AB}$, $\lambda = \text{const}$, $A, B = 0, ..., n$ yields

$$\lambda + \dot{H} + H_2 = 0 \tag{4.25}$$

and

$$\lambda + \dot{\alpha}_k + \alpha_k H = 0 \tag{4.26}$$

Summing up the n equations (4.26), we get

$$n\lambda + \dot{H} + H^2 = 0 \tag{4.27}$$

It holds: $a_k(t)/a_j(t)$ is constant iff $\beta_k = \beta_j$.

Let $B = \sum \beta_k^2$, then it holds: the metric (4.23) has $n \geq 2$ in the normal form iff $B \neq 0$. Proof: $B = 0$ iff $\beta_k = 0$ for every k. But equality of all the β_k implies $\beta_k = 0$ because of $\sum \beta_k = 0$. q.e.d.

Inserting $H_2 = B + H^2/n$ into eq. (4.25) we get

$$n\lambda + n\dot{H} + nB + H^2 = 0 \tag{4.28}$$

Eq. (4.26) can be written as

$$n\lambda + n\dot{\beta}_k + \dot{H} + n\beta_k H + H^2 = 0 \tag{4.29}$$

(Of course, summing up the n equations (4.29) we recover (4.27)). The difference between (4.27) and (4.28) leads to

$$\dot{H} = -\frac{n\,B}{n-1} \tag{4.30}$$

Because of $B > 0$ we have $\dot{H} < 0$.

The difference between (4.29) and (4.27) reads

$$\dot{\beta}_k + \beta_k H = 0 \tag{4.31}$$

This implies $\dot{B} = -2HB$ and with (4.30)

$$\left(\frac{\dot{B}}{B}\right)^{\cdot} = \frac{2n}{n-1} B \tag{4.32}$$

For $\lambda = 0$, we get from (4.27) $H = 1/t$, $(t > 0)$, i.e., with (4.30),

$$B = \frac{n-1}{n\;t^2} \tag{4.33}$$

With $H = 1/t$, we can solve eq. (4.31), obtaining $\beta_k = p_k/t$, $p_k =$ const. Together with eq. (4.33), we finally get

$$\sum p_k = 0 \qquad\qquad \sum p_k^2 = \frac{n-1}{n}$$

Consequently, $\alpha_k = \beta_k + 1/(nt) = q_k/t$, with

$$\sum q_k = \sum q_k^2 = 1 \tag{4.34}$$

One can integrate these equations to $a_k(t) = t^{q_k}$ and eqs. (4.34) give the defining condition for the d-dimensional Kasner solution.

Now we have to check under which conditions the Kasner solution

$$ds^2 = dt^2 + \sum_{k=1}^{n} t^{2q_k} (dx^k)^2 \tag{4.35}$$

is flat. (By the way, $n > 1$ and (4.34) already imply $B > 0$). However, this requires the calculation of the Weyl tensor.

IV.4. Calculating the Weyl tensor.

Now we take the metric (4.24), assume that it is already an Einstein space, and calculate the Weyl tensor with the notations used in sect. IV.3. In this section the sum conventions will not be used. It turns out that the non-trivial components of the Weyl tensor read

$$0 = C^{0k}{}_{0k} = -\dot{\alpha}_k - \alpha_k^2 + c_1 \tag{4.36}$$

and

$$0 = C^{jk}{}_{jk} = -\alpha_j \alpha_k + c_2 \qquad\qquad j \neq k \tag{4.37}$$

where c_1 and c_2 are constants which vanish iff ds^2 has vanishing curvature scalar. Now we are prepared to continue our discussion.

IV.5. Dimension $D \geq 4$. Third part.

Let $R = 0$, i.e. $c_2 = 0$ in eq. (4.37). This means that at most one of the functions α_j may be different from zero. If all of them are zero, we get eq. (4.1), if one of them is non-zero, then we get (4.2), so no new representations are found here.

Let $R \neq 0$, i.e. $c_2 \neq 0$ in eq. (4.37). Let j, k, l be three different indices (which is possible because $n \geq 3$). Then (4.37) implies $c_2 = \alpha_l \alpha_j = \alpha_l \alpha_k \neq 0$, hence $\alpha_l \neq 0$, and consequently $\alpha_j = \alpha_k$. Because of the arbitrariness in the choice of j and k, we conclude that all the functions α_k coincide. So we have the case $n = 1$ in the normal form which has already been discussed above. This concludes the discussion for the case that all the spaces $d\sigma_k^2$ are

flat. By condition D) (see sect.II), we may now assume that $d\sigma_1^2$ has non-vanishing curvature and consequently $d_1 \geq 2$. So we have to check under which circumstances (again, the index 1 will be omitted)

$$ds^2 = \pm a^2(t) d\sigma^2 + dt^2 \pm \sum_{k=2}^{n} a_k^2(t) d\sigma_k^2 \tag{4.38}$$

represents a space of constant curvature. If $a(t)$ is constant, this is impossible. Hence, $\alpha = \dot{a}/a \neq 0$. So, we have again the same situation as in sect. IV.2 and we may apply eq. (4.22) as follows: let

$$d\bar{s}^2 = \frac{1}{a^2(t)} \left[dt^2 \pm \sum_{k=2}^{n} a_k^2(t) d\sigma_k^2 \right] \tag{4.39}$$

then $d\bar{s}^2$ must be a space of non-vanishing constant curvature.

Let $D = 4$. Then necessarily $n = 2$, and $d\sigma_k^2$ is 1-dimensional, hence a flat space and $d\sigma^2$ has to be a plane of constant curvature. The results are easily calculated and are given by eqs. (4.12)-(4.19), where we have to insert $k = 2$ and $D = 4$ in eqs. (4.12) and (4.13).

Higher dimensions can be discussed by induction over D: eq. (4.39) has the same structure (up to a redefinition of t), and therefore we can deduce all possible metrics, from which we can calculate the corresponding metrics (4.38) of dimension at least higher by two.

However, this is quite involved, so we prefer a different approach. A more careful inspection of eq. (4.38) leads to the observation that eq. (4.37) remains valid. Therefore, the discussion at the beginning of sect. IV.5 still holds. Thus we can restrict to the case $n = 2$ in the following. So we have to check: $R = 0$

$$ds^2 = dt^2 \pm (a^2(t) d\sigma^2 + \delta_{\alpha\beta} dx^\alpha dx^\beta) \tag{4.40}$$

with $\dot{a} = 0$ and $d\sigma^2$ not flat.

$R \neq 0$

$$ds^2 = dt^2 \pm (a^2(t) d\sigma^2 + b^2(t) d\sigma_2^2) \tag{4.41}$$

with $\dot{a} \neq 0$, $\dot{b} \neq 0$, and $d\sigma^2$ not flat.

For the case $R = 0$, we get $a(t) = t$ and the metrics (4.12), (4.13). For the case $R \neq 0$, we get the generalization of (4.14)-(4.19) to higher dimensions, which read

$$n = 2, \qquad ds^2 = dt^2 + \sin^2 t \; d\Omega_j^2 + \cos^2 t \; d\Omega_k^2 \tag{4.42}$$

and

$$n = 2, \qquad ds^2 = dt^2 - \sin^2 t \; d\bar{\Omega}_j^2 - \cos^2 t \; d\bar{\Omega}_k^2 \tag{4.43}$$

for $R > 0$ and

$$n = 2, \qquad ds^2 = dt^2 + \sinh^2 t \; d\Omega_j^2 + \cosh^2 t \; d\bar{\Omega}_k^2 \tag{4.44}$$

and

$$n = 2, \qquad ds^2 = dt^2 - \cosh^2 t \; d\Omega_j^2 - \sinh^2 t \; d\bar{\Omega}_k^2 \tag{4.45}$$

for $R < 0$, with $j + k = D - 1$. In eq. (4.42), only $2 \leq j \leq k$ give new cases, these are $(D-4)/2$ for even D and $(D-3)/2$ for odd D. The same result appears for eq (4.43). In eq. (4.44), all j, $k \geq 2$ give new cases, $D - 4$ in sum, as does eq. (4.45).

V. A more general ansatz.

In this section, we consider a slightly different ansatz for the metric of a space of constant curvature, which is suitable for a direct determination of its general form from the solution of a simple system of differential equations. It is well-known, in fact, that a maximally symmetric space is uniquely characterized by the property that

$$R_{ijkl} = \lambda(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{5.1}$$

with a constant λ. Adopting an appropriate ansatz for the metric, (5.1) gives a set of differential equations which determine the conditions under which the space is of constant curvature.

We take a metric of the form

$$ds^2 = dt^2 \pm \left[a^2(t)d\rho^2 + \sum_{i=1}^{D-2} b_i^2(\rho, t)(dx^i)^2\right] \tag{5.2}$$

where the metric functions depend on t and one "spatial" coordinate ρ. A more general form of the ansatz would require that $a = a(t, \rho)$, but our choice is necessary in order to obtain a tractable set of differential equations from (5.1). Anyway, our choice is not restrictive since the metric (5.2) can be seen as a multi-warped product of a 2-dimensional space, spanned by the coordinates t and ρ, with a $(D-2)$-dimensional space, and is well known that the metric of a 2-dimensional space of constant curvature can always be put in the form $dt^2 \pm a^2(t)d\rho^2$ by a suitable choice of coordinates.

For simplicity, in this section we consider the case of Lorentzian signature. The Euclidean case is easily obtained following the same procedure. Inserting the ansatz (5.2) into (5.1), one obtains the following equations

$$\ddot{a} = \lambda a \tag{5.3}$$

$$\ddot{b}_i = \lambda b_i \tag{5.4}$$

$$\dot{a}\dot{b}_i - \frac{b_i''}{a} = \lambda a b_i \tag{5.5}$$

$$\dot{b}_i\dot{b}_j - \frac{b_i' b_j'}{a^2} = \lambda b_i b_j \tag{5.6}$$

$$a\dot{b}_i' - \dot{a}b_i' = 0 \tag{5.7}$$

where a dot denotes a derivative with respect to t and a prime a derivative with respect to ρ. Eq. (5.7) implies that

$$\frac{d}{dt}\left(\frac{b_i'}{a}\right) = 0 \tag{5.8}$$

Hence, either $b_i' = 0$ or $b_i = a(t)g_i(\rho)$. We can now pass to discuss the solutions of (5.3)-(5.7). In the following, we tacitly assume that the transformations of section II can be applied to change the form of the solutions to their normal form.

V.1. $\lambda > 0$.

If $\lambda > 0$, without loss of generality we can take $\lambda = 1$ by condition B). Eq. (5.3) then implies $a = Ae^t + Be^{-t}$ with A and B integration constants.

Moreover, if $b_i' = 0$, eq. (5.4) implies that $b_i = C_i e^t + D_i e^{-t}$, with C_i and D_i integration constants. Eqs. (5.5) and (5.6) then read

$$AD_i = -BC_i \qquad\qquad C_i D_j = -C_j D_i \tag{5.9}$$

Only two distinct classes of solutions are avalaible: if either $A = C_i = 0$ or $B = D_i = 0$, one has for any dimension D a solution of the general form (4.7). Further non-trivial solution are avalaible only in $D = 3$, when $D_1 = -BC_1/A$, or in $D = 2$ and, modulo the transformations of sect. II they reduce to the form (3.14) in $D = 3$ or (3.6), (3.7) in $D = 2$.

If $b_i' \neq 0$, instead, one has

$$b_i = A(t)g_i(\rho) = (Ae^t + Be^{-t})g_i(\rho)$$

which substituted in (5.5), (5.6) gives

$$g_i'' = -4ABg_i \qquad\qquad g_i' g_j' = 4ABg_i g_j \tag{5.10}$$

One must distinguish the case $AB > 0$ from $AB < 0$. If $AB > 0$, $a(t)$ can be reduced to $\cosh t$, while, after a rescaling of ρ, $g_i = F_i \sin\rho + G_i \cos\rho$ and (5.10) implies $F_i F_j + G_i G_j = 0$. A solution to the last equation is available only for $D \leq 4$, and can be put in the form[3]

$$ds^2 = dt^2 - \cosh^2 t\ (d\rho^2 + \sin^2\rho\ dx^2 + \cos^2\rho\ dy^2) \tag{5.11}$$

which is of the type (4.11).

If $AB < 0$, instead, $a(t)$ can be reduced to $\sinh t$, while $g_i = F_i e^\rho + G_i e^{-\rho}$ and (5.10) implies $F_i G_j + G_i F_j = 0$. A solution to this equation valid in any dimensions, is given by $F_i = 0$ or $G_i = 0$ and can be written

$$ds^2 = dt^2 - \sinh^2 t\ (d\rho^2 + e^{2\rho} \sum dx_k^2) \tag{5.12}$$

Another solution is avalaible when $D \leq 4$, if $F_2 G_1 = -F_1 G_2$, which can be reduced to the form

$$ds^2 = dt^2 - \sinh^2 t\ (d\rho^2 + \sinh^2\rho\ dx^2 + \cosh^2\rho\ dy^2) \tag{5.13}$$

[3]Here and in the following, lower dimensional solutions can be obtained by suppressing some of the coordinates x, y...

Both solutions (5.11) and (5.12) are special cases of (4.9). A last possibility remains to be investigated, namely the case in which some of the b_i are independent of ρ and some are not. It is easy to see that in this case, eq. (5.6) is a consequence of (5.5) and hence one gets two new 5-dimensional solutions, and one in arbitrary dimensions, by combining (5.11)-(5.13) with (3.14)

$$ds^2 = dt^2 - \sinh^2 t\ (d\rho^2 + e^{2\rho} \sum dx_k^2) - \cosh^2 t\ dy^2 \tag{5.14}$$

$$ds^2 = dt^2 - \sinh^2 t\ (d\rho^2 + \sinh^2 \rho\ dx^2 + \cosh^2 \rho\ dy^2) - \cosh^2 t\ dz^2 \tag{5.15}$$

$$ds^2 = dt^2 - \cosh^2 t\ (d\rho^2 + \sin^2 \rho\ dx^2 + \cos^2 \rho\ dy^2) - \sinh^2 t\ dz^2 \tag{5.16}$$

The solutions (5.14)-(5.16) are all special cases of (4.45).

V.2. $\lambda < 0$**.**

If $\lambda < 0$, (5.3) yields $a = A\sin t + B\cos t$, with A and B integration constants. If $b_i' = 0$, eq. (5.4) is solved by $b_i = C_i \sin t + D_i \cos t$ and eqs. (5.5) and (5.6) imply

$$AC_i = -BD_i \qquad\qquad C_iC_j = -D_iD_j \tag{5.17}$$

These relations admit solutions only for $D = 3$, for example $B = C_1 = 0$ or $A = D_1 = 0$. All these solutions are equivalent, via the transformations of section II, to (3.11).

If $b_i' \neq 0$, instead, one has

$$b_i = A(t)g_i(\rho) = (A\sin t + B\cos t)g_i(\rho)$$

which substituted in (5.5), (5.6) give respectively,

$$g_i'' = -(A^2 + B^2)g_i \qquad\qquad g_i'g_j' = -(A^2 + B^2)g_ig_j \tag{5.18}$$

These equations have solution $g_i = F_ie^{\rho} + G_ie^{-\rho}$, with $F_iG_j + G_iF_j = 0$. Solutions to this equation exist up to 4 dimensions and take the form

$$ds^2 = dt^2 - \sin^2 t\ (d\rho^2 + \sinh^2 \rho\ dx^2 + \cosh^2 \rho\ dy^2) \tag{5.19}$$

and hence are of the type (4.6).

Finally, in the case in which some of the b_i are independent of ρ and some are not, it is easy to see that eq. (5.10) is always satisfied and hence one gets one new 5-dimensional solution, which again is a special case of (4.43)

$$ds^2 = dt^2 - \sin^2 t\ (d\rho^2 + \sinh^2 \rho\ dx^2 + \cosh^2 \rho\ dy^2) + \cos^2 t\ dz^2 \qquad (5.20)$$

V.3. $\lambda = 0$.

If $\lambda = 0$, a and b must be linear in t. Proceeding as before, one obtains the following possibilities: if $a = t$, b_i is constant for any i or can have the form $b_1 = t \sinh \rho$, $b_2 = t \cosh \rho$. One then recovers the solutions (4.2), or

$$ds^2 = dt^2 - t^2(d\rho^2 + \sinh \rho^2\ dx^2 + \cosh^2 \rho\ dy^2) \qquad (5.21)$$

which is equivalent to (3.19). If a is constant, one can have $b_1 = \rho$, $b_i =$ const, $i \neq 1$ and hence (4.13).

To conclude this section, we observe that no new solutions are introduced by the ansatz (5.2). However, we did not formally define which is the generalization of condition F), and so in this sect. V, the classification is not such a strict one as in sect. IV. This fact may lead to the conjecture that even assuming a more general dependence of the metric on the "spatial" coordinates, no new forms can be obtained for the solutions, in addition to those listed in the previous sections.

VI. Discussion.

Now we have finished the classification of the spaces of constant curvature, and it is useful to sum up potential applications of this. First, consider any type of cosmic no-hair theorem stating that under certain circumstances every solution of the Einstein field equation is asymptotically a space of constant curvature. To find out regions where this theorem applies it is necessary to know which spacetimes of a given class, e.g. the multidimensional warped products we considered here, is exactly a (generalized) de Sitter spacetime. We found unexpected forms on the one hand, and on the other hand, it came out that the cartesian product between a four-dimensional de Sitter

spacetime and one or more static internal spaces is never a space of constant curvature .

Example: metric (4.19) is a metric for a cosmological model of Bianchi type III. Usually, Bianchi type III consists of anisotropic models only, and so (4.19) may be used to test the range of validity of the no-hair theorem within the class of Bianchi type III models, which seems not to have been done up to now.

This list may help identifying "new solutions" of Einstein's field equations with or without cosmological term in an arbitrary number of dimensions. The most remarkable result of our paper seems to be that no metric with more than two factors is possible. If n is the number of factors, we have found for every $D \geq 2$ twelve representations with $n = 1$ and $5D - 10$ representations with $n = 2$ if D is even and $5D - 9$ representations with $n = 2$ if D is odd.

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References

??1] U. Kasper, M. Rainer and A. Zhuk, Gen. Rel. Grav. **29**, 1123 (1997).

??2] V. D. Ivashchuk and V. N. Melnikov, Grav. and Cosmol. **2**, 211 (1996).

??3] U. Bleyer, D.-E. Liebscher, H.-J. Schmidt and A. Zhuk, Wiss. Z. Univ. Jena **39**, 20 (1991).

??4] H.-J. Schmidt, Fortschr. Phys. **41**, 179 (1993).

??5] J. Bičak and J. Griffiths, Ann. Phys. (NY) **252**, 180 (1996).

??6] M. Cadoni and S. Mignemi, Mod. Phys. Lett. A **10**, 367 (1995).